operators of this order contained in such a subgroup is $p^{\alpha_1+\alpha_2} - p^{\alpha_1+\alpha_2-1}$. The number of these subgroups is therefore the quotient of the given numbers of operators since the numbers of ways in which the smaller independent generator can be chosen after the larger independent generator has been selected is the same for the group as for the subgroup.

Hence the number of the subgroups in which the two independent generators are of orders p^{α_1} and p^{α_2} is $p + 1$ times $p^{\alpha_2-\alpha_1-1}$. The exponent $\alpha_2 - \alpha_1 - 1$ cannot exceed $\beta - 2$ and has this value only when $\alpha_2 = \beta$ and $\alpha_1 = 1$. There are two cases in which this exponent is $\beta - 3$, viz., when $\alpha_2 = \beta$ and $\alpha_1 = 2$ or when $\alpha_2 = \beta - 1$ and $\alpha_1 = 1$. As similar considerations apply to the other cases the number of these subgroups is obviously given by the formula

$$(p + 1)(p^{\beta-2} + 2p^{\beta-3} + \ldots\ldots + (\beta - 1)p^0)$$

whenever $\beta > 1$. In this case the total number of the subgroups in the abelian group of order p^m whose independent generators are of orders p^β, p^γ is therefore given by the formula

$$\beta + (p + 1)(p^{\beta-2} + 2p^{\beta-3} + \ldots + (\beta - 1)p^0) + (\gamma - \beta)(p^\beta - 1)/(p - 1).$$

The entire group is included among the subgroups in the enumeration by this formula. When $\beta = 1$ the first and the last term of this formula, taken together, give the total number of the subgroups in question.

THE TRANSFORMATION OF A LAGRANGIAN SERIES INTO A NEWTONIAN SERIES

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A series of type

$$\sum_{n=0}^{\infty} a_n \frac{(z - c - 1)(z - c - 2) \ldots (z - c - n)}{(z + c + 1)(z + c + 2) \ldots (z + c + n)} = \sum a_n R_n(z) \quad (1)$$

will be called a Lagrangian series on account of the extensive investigations of René Lagrange[1] relating to the expansion of functions in series of this and allied types.

A transformation of a Newtonian series into a series of type (1) may be effected with the aid of Lagrange's expansion

$$p(z + c/, p) = \Sigma\,(2c + 2n + 1)(n + p - 1/, p - 1)\times (2c + n/, p - 1)R_n(z) \quad (2)$$

where the symbol $(q/, p)$ is used for the coefficient of t^p in $(1 + t)^q$. When $c = 0$ the transformation of the series (1) into a Newtonian series may be based on the formula

$$\frac{(1 - z, p)}{(1 + z, p)} = \sum_{n=p+1}^{\infty} (-)^{n + p - 1}\,(z/, n)\,\frac{(1 - n, p)}{(1 + n, p)} \quad (3)$$

where $(m, n) = m(m + 1)(m + 2)\ldots(m + n - 1)$, $(m, 0) = 1$, $(m, 1) = m$.

To prove this formula we observe that the series on the right may be written in the form

$$z(1 - z)(2 - z)\ldots(p - z)p\,F(p + 1, 1 + p - z;\ 2p + 2;\ 1)/(2\,p + 1)! \quad (4)$$

When $R(z) > 0$ the hypergeometric series may be summed by the formula of Gauss and has the value

$$(2p + 1)!\,/z(z + 1)(z + 2)\ldots(z + p)p!$$

We thus obtain the expression on the left.

The relation (3) gives rise to a number of interesting expansions of which we mention three

$$F(1 - z; 1 + z; x) = \sum_{n=1}^{\infty} (-)^{n-1}\,(z/, n)F(1 - n; 1 + n; -x)$$

$$F(a, 1 - z; 1 + z; x) = \sum_{n=1}^{\infty} (-)^{n-1}\,(z/, n)F(a, 1 - n; 1 + n; -x) \quad (5)$$

$$F(1 - z; 1 + z, a; x) = \sum_{n=1}^{\infty} (-)^{n-1}\,(z/, n)F(1 - n; 1 + n, a; -x).$$

In the second of these we suppose that $R(a)$ is negative when $x = -1$; otherwise we must suppose that $|x| < 1$. In the former case we may use Gauss's formula for the hypergeometric series and obtain the identity

$$F(a, 1 - z; 1 + z; -1) = zF(1 - z, 1 - {}^1/_2a, {}^3/_2 - {}^1/_2a; {}^3/_2, 2 - a; 1) \qquad R(a) < 0 \quad (6)$$

which is certainly true when $a = -1$. It is also true when $a = 0$; When $c = 0$ the expansion of a function $f(z)$ in a series of type (1) may be written in the symbolical form

$$f(z) = \frac{1}{z} f(1) + \sum_{n=1}^{\infty} (-)^n \frac{(z-1)(z-2)\ldots(z-n)}{z(z+1)(z+2)\ldots(z+n)} \times$$
$$(2n+1)P_n(1-2E)f(1) \quad (7)$$

where E^s is an operator which changes $f(1)$ into $f(s+1)$. When the operations are performed the series gives a formula of interpolation. A formula something like this has been mentioned by E. Hille.[2] When $f(z)$ is a function represented by a Laplacian integral

$$f(z) = \int_0^{\infty} e^{-zt} g(t)dt \quad (8)$$

in which $g(t)$ is continuous we obtain a confirmation of Lerch's result[3] that when $F(1), f(2), f(3) \ldots$ are all zero $f(z)$ is zero for all values of z exceeding unity.

An interesting example of the formula of interpolation (7) is furnished by the equation

$$\frac{uz}{u+z-1} = F(1, {}^3/_2, 1-u, 1-z; {}^1/_2, 1+u, 1+z; 1) \quad (9)$$

the generalized hypergeometric series being convergent when $R(u+z) > 1$. Converting this series into a Newtonian series by means of the relation (3) we obtain the equation

$$\frac{uz}{u+z-1} = \sum_{n=1}^{\infty} (-)^{n-1} (z/, n) F(1, {}^3/_2, 1-u, 1-n; {}^1/_2, 1+u, 1+n; -1). \quad (10)$$

Calculating the coefficients in the Newtonian series by the usual rule we find that

$$F(1, {}^3/_2, 1-u, 1-n; {}^1/_2, 1+u, 1+n; -1) = n!/(u+1, n-1). \quad (11)$$

The hypergeometric series on the left is "nearly-poised" in the terminology of F. J. Whipple.[4] The identity (6) is a particular case of Whipple's identity

$$\Gamma({}^1/_2k - b)\,\Gamma(k)\,F(a, b; k-b; -1) = \Gamma(k-b)\,\Gamma({}^1/_2k) F(b, {}^1/_2k - {}^1/_2a,$$
$${}^1/_2 + {}^1/_2k - {}^1/_2a; k-a, {}^1/_2 + {}^1/_2k; 1) \quad (12)$$

which is an extension of one due to Kummer. The relation (11) is a particular case of Whipple's identity[5] $F(a, 1 + {}^1/_2a, c, d; {}^1/_2a, 1+a-c, 1+a-d; -1) = F(c, d; 1+a; 1)$.

Another interesting example of the formula (7) is obtained by writing (9) in the form

$$1/(u + z - 1) = \sum_{n=0}^{\infty} (2n + 1)G_n(u)G_n(z), \tag{13}$$

where $$z(1 + z, n)G_n(z) = (1 - z, n).$$

With this notation the Legendre series associated with $(^1/_2 - {}^1/_2 s)^{z-1}$ is

$$(^1/_2 - {}^1/_2 s)^{z-1} \sim \sum_{n=0}^{\infty} (2n + 1)P_n(s)G_n(z) \tag{14}$$

and a related series is

$$\int_0^{\infty} e^{-xt} (1 + t)^{-z}\, dt = \sum_{n=0}^{\infty} (2n + 1) V_n(x)G_n(z), \tag{15}$$

where $$V_n(x) = \int_0^{\infty} e^{-xt} P_n[(t - 1)/(t + 1)]dt/(t + 1).$$

If $R(z) > 0$ there is a further relation

$$t^{z-1} / \Gamma(z) = \sum_{n=0}^{\infty} (2n + 1)Z_n(t)G_n(z), \tag{16}$$

where $Z_1(t) = F(-n, n + 1; 1, 1; z)$. Also, if $R(z) > 1$, $0 < u \leq 1$

$$\Gamma(u + z - 1)/\Gamma(u)\,\Gamma(z) = \sum_{n=0}^{\infty} (2n + 1)F_n(2u - 1)G_n(z)$$

where $F_n(2u - 1) = F(-n, n + 1, u; 1, 1; 1)$.

[1] R. Lagrange, *Acta Mathematica*, **64**, 1–80 (1935).
[2] E. Hille, *Compositio Mathematica*, **6**, 93–102 (1938).
[3] M. Lerch, *Acta Mathematica*, **27** (1903).
[4] F. J. Whipple, *Proc. Lond. Math. Soc.* (2), **26**, 257–272 (1927).
[5] F. J. Whipple, *Ibid.* (2), **24**, 247–263 (1925).